***A Mathematical History of Division in Extreme and Mean Ratio*. By Roger Herz-Fischler. Waterloo, Ontario: Wilfrid Laurier University press, 1987. xvi + 191 pp.**

Published in *Historia Mathematica* **17** (1990), 175–178

As stated in the title, the book tells the *mathematical* history of division in extreme and mean ratio (DEMR) from the beginnings through the 18th century. The non-mathematical history of the same famous ratio, under which aspect it is most often called by its *alias* "the Golden Number", is in reserve for another book. The mathematical history, on the other hand, includes both the theory and construction of the ratio itself and constructions which make use (or *might* make use) of the division. The book is thus also a history of constructions and discussions of the pentagon, the decagon, the icosahedron and the dodecahedron.

The earliest extant source mentioning DEMR is Euclid's *Elements*. It is therefore fully appropriate that the book starts with a detailed exposition of all propositions and definitions from the *Elements* somehow related to the DEMR or its applications. After a chapter on mathematical topics essential to the discussion (concepts relating to the question of "geometrical algebra", "side and diagonal numbers", incommensurability, and the "Euclidean algorithm"), the author goes back in time to see how much can be inferred about the pre-Euclidean history of the concept. Next comes an (apparently exhaustive) presentation of constructions, discussions and calculations related to DEMR from Archimedes to Mascheroni. Finally, a first appendix arrays the variety of names given to DEMR, while a second appendix quotes the opinions of Campanus, Pacioli, Ramus, Kepler and others concerning its marvellous power.

Up to the time of the *Elements* (c. 300 B.C.), the history of DEMR has to be based on indirect evidence. Since the mid-nineteenth century, a wealth of theories, hypotheses and speculations have grown from this absence of constraints on scholarly imagination. Herz-Fischler offers a broad and liberal survey that includes all serious attempts to deal with the subject (and a few less serious), and then discusses these various interpretations in the light of available evidence. Many properties of DEMR are easily demonstrated by means of the pentagram, which sources from later Antiquity connect with the Pythagorean order. Therefore, Herz-Fischler goes through the early occurrences of the pentagram, in ostraca, graffiti, and coins. He argues convincingly that prehistoric and Bronze Age pentagrams and an Old Babylonian calculation of the area of a pentagon have nothing to do with the matter, and, what is more important, that a figure in such widespread use (be it for decoration or with some hypothetical symbolic meaning) could hardly have been utilized as a specific symbol of recognition by the early Pythagoreans. He demonstrates, moreover, that there is no particular reason to believe (but often good reason to doubt) the late sources ascribing the application of areas and the *construction* of the regular pentagon and dodecahedron to "the Pythagoreans". Similarly, the theories connecting DEMR and the regular pentagon to the discovery of incommensurability are shown to rest on insecure foundations. Finally, in the same vein, strong arguments are set forth that the "section" on which Eudoxos is claimed by Proclos to have contributed a number of propositions was *not* the section obtained by DEMR.

In all cases, Herz-Fischler sets forth these earlier theories with great loyalty before divulging his criticisms. In the first half of the book much more space is therefore devoted to the presentation of problems and to these theories than to describing the author's own preferred scenario. But, of course, the author has his own convictions regarding the subject. These convictions are at considerable variance with all conventional wisdom concerning the early history of Greek geometry, but they are mostly well argued – cf. below on one of the points which in the reviewer's opinion are less well so). The argument is complex and interwoven with the critical exposition of other positions and cannot be summarized adequately in a review. Central components, however, are derived from an analysis of the *Elements*, and in particular from the double occurrence of DEMR: in "area formulation" and without the name in II.11, and then in "ratio formulation" and with the name in book VI (def. 3 and prop. 30). The first formulation is argued to be the original one, and to have arisen inside a "research program" aimed at the construction of regular polygons, probably in the early fourth century B.C.; this program will eventually have resulted in the construction of the regular pentagon and decagon. A second and slightly later program, aiming at the inscription of regular polyhedra, will have resulted in the construction of the dodecahedron and the icosahedron and in the remaining results of *Elements* XIII – still in area formulation. Theaetetos is suggested to have launched both programs.

This scenario has implications for the origin of the area geometry of *Elements* II (the presumed "geometric algebra") and for that of the classification of irrationals (*Elements* X). According to Herz-Fischler, prop. II.6 is (with II.11) a result of the "first program"; *Elements* II is believed to be a compilation of superficially similar lemmas needed elsewhere (in- or outside the *Elements*) and not a theoretically connected construct, while the comparable propositions XIII.1-5 would be a series of *ad hoc* lemmas to be used in the remainder of book XIII. The classification of irrationals is regarded as a spin-off originating from the "first program", having been formulated perhaps originally inside the framework of a pre-Eudoxean theory of proportion.

Not all parts of Herz-Fischler's scenario are equally compelling; quite a few arguments are merely plausible conjectures, and at times the plausibility depends critically upon problematic interpretations. One particular point that troubled the reviewer concerns the understanding of Babylonian "algebra", which builds exclusively and uncritically upon Solomon Gandz's purely numerical reading of the texts [Gandz 1937 (not 1938 as stated)]. Indeed, if the Babylonian texts are read as "naive geometry" (for which reading, see, e.g., [H¢yrup 1989]), the whole question of *Elements* II must be raised anew; so, propositions II.9 and II.10, which are never used in the *Elements* and thus hardly included in Book II as useful lemmas, are identical with problems 8 and 9 from the Old Babylonian tablet BM 13901. (In fairness, it should be noticed that Gandz's interpretation had not been challenged in 1982, when the book under review was practically finished). Yet, even if parts of Herz-Fischler's scenario should fall others may well stand securely, and everybody working on the early history of Greek geometry will benefit both from the critical survey of older work and from the fresh and unconventional conjectures.

The pre-Euclidean origins of DEMR can only be recovered by indirect arguments, for which reason they have attracted throngs of workers. The post-Euclidean history is much better documented in accessible sources; one may wonder whether the constraints thereby imposed upon historical inventiveness explain why nobody ever worked it up systematically before Herz-Fischler. In any case, the "post-Euclidean« part of Herz-Fischler's book is much more descriptive than the "pre-Euclidean" and "Euclidean" chapters. Of course, there are still open problems, not least as regards the interpretation of certain Ancient sources; most important is a discussion of "Book XIV of the *Elements*". Still, on the whole, this part is a useful overview of the references to and uses of DEMR (or absence of references) in Ancient, in Medieval Islamic, Indian, Chinese and European and in Early Modern European mathematics. Among the topics traced are the use of DEMR and related results for the computation of trigonometric functions of 36° and 72°, and the possible connection between certain recurrent algebra problems (among which the problem "$x+y=10$; $x^2=10\cdot y$") and DEMR, and of course the regular pentagon and decagon and the dodecahedron and icosahedron. In India nothing relevant turns up before Bhaskara II, and in China nothing at all. Most important among the mathematicians discussed in this part are Hero; Ptolemy; Pappos; al-Khwa→rizm♪; Abu→ Ka→mil; al-B♪ru→n♪; Leonardo Fibonacci; Piero della Francesca; Luca Pacioli; Cardano; Bombelli; and Kepler.

A commendable feature of the book is the extensive use of quotations from Ancient philosophers, commentators and others, mostly taken from established translations. In a few cases this leads to unfortunate results because the original translators did not have Herz-Fischler's specific problems in view. thus, for example, a quotation from Proclos' *On Euclid I* (p. 67) follows Morrow and ascribes a point of view to "Eudemus and his school", while the ensuing arguments would be much better served by the imprecision of Proclos's own "The circle around Eudemos". On p. 50, Herz-Fischler tries to get around a similar problem by emending (tacitly) Harold Fowler's translation of a key passage from Plato's *Theatetus* 147D, replacing »roots« by »'dy→namis'« (singular, not plural as required); but the inconsistency in Fowler's translation is conserved, since "squares" in the same line (which translates the same Greek word) is conserved.

These, however, are minor problems. So are certain other technical deficiencies. The first of these concerns the proofreading of diagrams and appurtenant texts. In quite a few cases, letters used in the text are forgotten in the diagrams; in others they do not correspond. Luckily, however, the reader can easily repair most of these errors himself or find the correct letterings in the corresponding diagrams in Heath's translation of the *Elements*. The only error of this kind which caused me some moments of reflection is equation (1) on p. 100, whose right-hand side should read $\{OA^2+AD^2\}:AD^2=\{(CO+OA)^2+CA^2\}:CA^2$ instead of the concoction $\{(CO+OA)^2+AD^2\}:AD^2$.

A second technical problem also has to do with the diagrams. Very often these are drawn grossly out of proportion. In a few cases this may have been done for pedagogical reasons, but often the distortions are downright misleading. It required all the reviewer’s concentration to conceptualize 11 mm as the half of 28.4 mm or to see a line divided in the ratio 1:2.6 as being "really" divided in extreme and mean ratio (1:1.61...) – to name but two examples, both to be found on p. 30 (fig. I-25 and I-26). Computers may make nice drawings, but they seem still to be in need of some supervision.

These minor deficiencies are balanced by major merits which the author has achieved in intentional reaction to the "obscure bibliographical references; as well as incorrect translations, incorrect inferences from quotations, and misrepresentation of the mathematical process actually involved in the original" abunding in the literature on this no less than other subjects (p. xi). Firstly, Herz-Fischler’s argument is always clear, and clearly arranged. Secondly, the book as a whole is well organized. Thirdly and finally, through his world-wide hunt for information on DEMR and for secondary literature touching on the subject the author has accumulated a veritable profusion of references. The contents of the bibliography will be useful to every scholar working in the vicinity of DEMR, and the fullness of the information given will be appreciated by every interlibrary service.

REFERENCES

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